3.2 Verify Properties of Triangles

Since triangular frames are strong and simple to make, they are widely used to strengthen buildings and other structures. This section applies analytic geometry to verify the properties of specific triangles.

Investigate

Tools
- grid paper

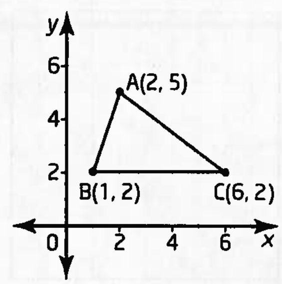
How can you verify the properties of an isosceles triangle?

Method 1: Use Pencil and Paper

1. Draw the triangle with vertices A(2, 5), B(1, 2), and C(6, 2). Use the coordinates of the vertices to verify that △ABC is isosceles. Which angles of △ABC are equal?

2. Describe how you could fold your drawing of △ABC to confirm that it is isosceles.

3. Find the coordinates of the midpoint, D, of side AB. Draw the median from vertex C. Use slopes to verify that this median is perpendicular to AB.



4. Explain how you can use congruent triangles to verify that CD is perpendicular to AB. Describe two different ways to show that △ADC and △BDC are congruent.

5. Use angle sums to verify that CD bisects ∠ACB.

6. Describe another way to show that CD bisects ∠ACB.

7. Reflect How are the median, the altitude, and the angle bisector at vertex C related? How are these line segments related to the right bisector of AB? Do you think all isosceles triangles have these properties? Explain your reasoning.

Tools

- computer with *The Geometer's Sketchpad®*

Technology Tip

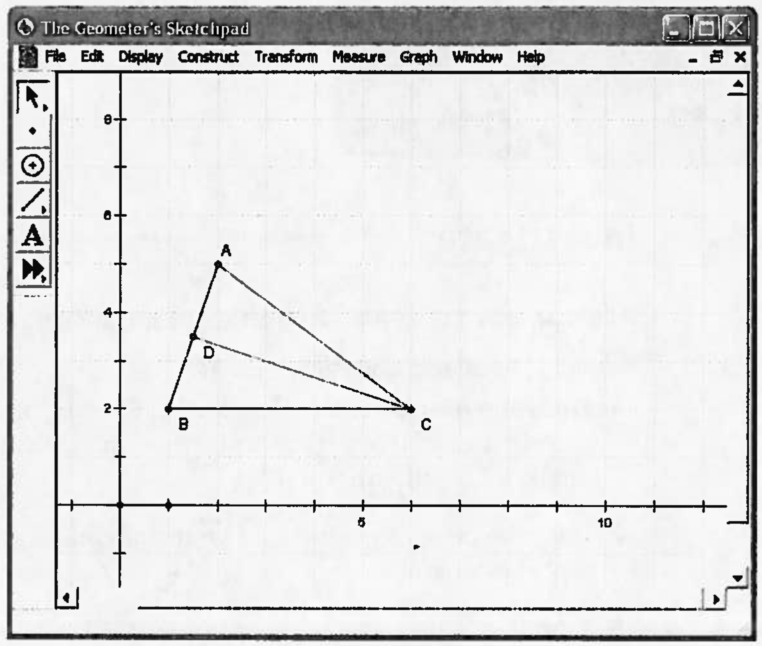
You can also place the vertices by choosing **Plot Points** from the **Graph** menu and typing the coordinates.

Method 2: Use *The Geometer's Sketchpad®*

1. Choose **Show Grid** from the **Graph** menu. Then, choose **Snap Points**. Move the origin so that most of the work area is in the first quadrant. Construct the triangle with vertices A(2, 5), B(1, 2), and C(6, 2).

2. Measure the length of each side to verify that △ABC is isosceles. Which angles of △ABC are equal?

3. Construct the midpoint, D, of side AB. Then, construct the median from vertex C. Measure ∠ADC. What can you conclude about the median CD?



4. Measure and compare ∠ACD and ∠BCD.

5. Turn off **Snap Points**. Drag a vertex of △ABC to a new location. Move one of the other vertices until AC = BC. Observe the measures of ∠ADC, ∠ACD, and ∠BCD as you repeat this process for various locations of the vertices.

6. **Reflect** How are the median, the altitude, and the angle bisector at vertex C related? How are these line segments related to the right bisector of AB? Do you think all isosceles triangles have these properties? Explain your reasoning.

Tools

- TI-83 Plus or TI-84 Plus graphing calculator

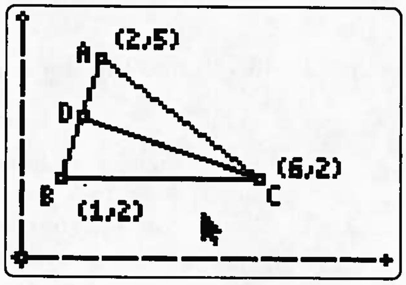
Method 3: Use a Graphing Calculator

1. Start the Cabri® Jr. application. If the axes are not displayed, choose **Hide/Show** from the **F5** menu; then, choose **Axes** from the submenu. Move the origin so that most of the work area is in the first quadrant.

2. Draw the triangle with vertices A(2, 5), B(1, 2), and C(6, 2). Choose **Coord. & Eq.** from the **F5** menu. Then, select the vertices to display their coordinates. To adjust a vertex, move the cursor to it, press ALPHA, and use the arrow keys to reposition the vertex.

3. Choose **Measure** from the **F5** menu, and then choose **D. & Length.** Measure the length of each side to verify that the triangle is isosceles. Which angles of $\triangle ABC$ are equal?

4. Choose **Midpoint** from the **F3** menu and construct the midpoint, D, of side AB. Then, choose **Segment** from the **F2** menu and construct the median from vertex C. Measure $\angle ADC$. What can you conclude about the median DC?



5. Measure and compare $\angle ACD$ and $\angle BCD$.

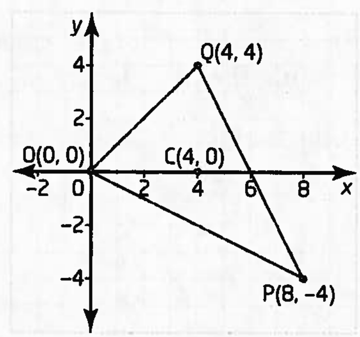
6. Drag a vertex of $\triangle ABC$ to a new location. Move one of the other vertices until AC = BC. Observe the measures of $\angle ADC$, $\angle ACD$, and $\angle BCD$ as you repeat this process for various locations of the vertices.

7. Reflect How are the median, the altitude, and the angle bisector at vertex C related? How are these line segments related to the right bisector of AB? Do you think all isosceles triangles have these properties? Explain your reasoning.

Example 1 Centroid of a Triangle

a) Verify that C(4, 0) is the centroid of $\triangle OPQ$.

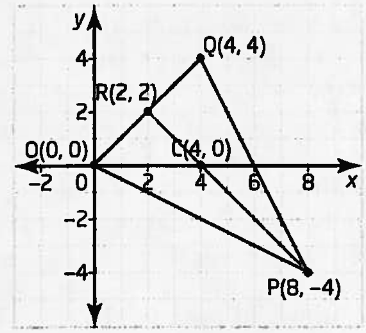
b) Verify that the centroid divides each median in a 2:1 ratio.



Solution

a) The centroid of a triangle is the point of intersection of the three medians. Verify that C(4, 0) is the centroid by showing that the coordinates of this point satisfy the equations for the lines that include the three medians.

Start by using the midpoint formula to find the coordinates of the midpoint, R, of side OQ.



$$\begin{aligned} R(x, y) &= \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right) \\ &= \left(\frac{0 + 4}{2}, \frac{0 + 4}{2}\right) \\ &= (2, 2) \end{aligned}$$

Use these coordinates to find the slope of the median from vertex P(8, −4).

$$\begin{aligned} m_{PR} &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{2 - (-4)}{2 - 8} \\ &= \frac{6}{-6} \\ &= -1 \end{aligned}$$

I could also substitute the coordinates of points P and R into $y = mx + b$ to get two equations with m and b as the unknowns. Solving this system of equations gives values for m and b.

Use this slope and the coordinates of point R to find the y-intercept.

$y = mx + b$
$2 = -1(2) + b$
$2 = -2 + b$
$4 = b$

An equation for the line that includes the median PR is $y = -x + 4$.

Now, substitute the coordinates of point C(4, 0) into each side of the equation.

L.S. $= y$
$= 0$

R.S. $= -x + 4$
$= -4 + 4$
$= 0$

Since the coordinates of point C satisfy the equation and the point lies within the triangle, C(4, 0) lies on the median PR.

Next, find the midpoints, S and T, of sides OP and PQ.

$$\begin{aligned} S(x, y) &= \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right) \\ &= \left(\frac{0 + 8}{2}, \frac{0 + (-4)}{2}\right) \\ &= (4, -2) \end{aligned}$$

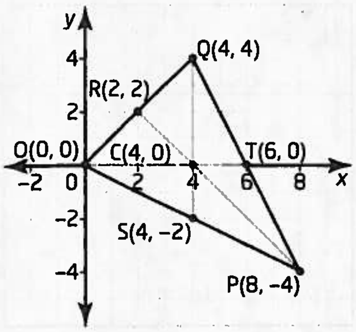
$$\begin{aligned} T(x, y) &= \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right) \\ &= \left(\frac{8 + 4}{2}, \frac{-4 + 4}{2}\right) \\ &= (6, 0) \end{aligned}$$

The method used for the median PR will also show that point C lies on the medians OT and QS. However, examining the coordinates of the points and the diagram of △OPQ reveals a shortcut for the medians OT and QS.

Since S(4, −2), C(4, 0), and Q(4, 4) all have the same x-coordinate, these three points lie on the vertical line with equation $x = 4$. Therefore, point C lies on the median QS. Similarly, O(0, 0), C(4, 0), and T(6, 0) are **collinear** since these points all lie on the x-axis. So, point C also lies on the median OT.

collinear

- lying on the same line



Since the point C(4, 0) lies on all three medians, it is the centroid of △OPQ.

b) To find the ratio of the parts of a median on either side of the centroid, use the length formula to find the length of each part.

For the median PR, compare the lengths of PC and RC.

$$\begin{aligned} PC &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ &= \sqrt{(8 - 4)^2 + (-4 - 0)^2} \\ &= \sqrt{4^2 + (-4)^2} \\ &= \sqrt{4^2 \times 2} \\ &= \sqrt{4^2} \times \sqrt{2} \\ &= 4\sqrt{2} \end{aligned}$$

$$\begin{aligned} RC &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ &= \sqrt{(2 - 4)^2 + (2 - 0)^2} \\ &= \sqrt{2^2 + 2^2} \\ &= \sqrt{2^2 \times 2} \\ &= \sqrt{2^2} \times \sqrt{2} \\ &= 2\sqrt{2} \end{aligned}$$

To move a factor out from under a square root sign, I take the square root of the factor:

$\sqrt{nx} = \sqrt{n} \times \sqrt{x}$

PC is twice the length of RC.

Since the median OT is a horizontal line segment, find the lengths of OC and TC by simply comparing the x-coordinates of the endpoints. Subtract the lesser x-coordinate from the greater one.

$$\begin{aligned} OC &= x_2 - x_1 \\ &= 4 - 0 \\ &= 4 \end{aligned} \qquad \begin{aligned} TC &= x_1 - x_2 \\ &= 6 - 4 \\ &= 2 \end{aligned}$$

Since QS is a vertical line segment, compare y-coordinates to find the lengths of QC and SC.

$$\begin{aligned} QC &= y_1 - y_2 \\ &= 4 - 0 \\ &= 4 \end{aligned} \qquad \begin{aligned} SC &= y_2 - y_1 \\ &= 0 - (-2) \\ &= 2 \end{aligned}$$

The centroid C(4, 0) divides each of the medians of △OPQ into a 2:1 ratio.

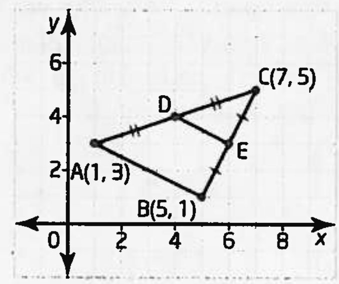
Example 2 Midpoints of the Sides of a Triangle

In △ABC, D is the midpoint of side AC and E is the midpoint of side BC.

a) Verify that line segment DE is parallel to side AB.

b) Verify that line segment DE is half the length of side AB.

c) Use geometry software to check your calculations in parts a) and b).



Solution

a) First, use the coordinates of the vertices to find the coordinates of the midpoints D and E.

$$\begin{aligned} D(x, y) &= \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right) \\ &= \left(\frac{1 + 7}{2}, \frac{3 + 5}{2}\right) \\ &= (4, 4) \end{aligned} \qquad \begin{aligned} E(x, y) &= \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right) \\ &= \left(\frac{5 + 7}{2}, \frac{1 + 5}{2}\right) \\ &= (6, 3) \end{aligned}$$

Now, use the coordinates of points A, B, D, and E to compare the slope of AB to the slope of DE.

$$\begin{aligned} m_{AB} &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{1 - 3}{5 - 1} \\ &= \frac{-2}{4} \\ &= -\frac{1}{2} \end{aligned} \qquad \begin{aligned} m_{DE} &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{3 - 4}{6 - 4} \\ &= -\frac{1}{2} \end{aligned}$$

Since the slopes are the same, AB is parallel to DE.

b) Use the distance formula to compare the length of DE to the length of AB.

$$\begin{aligned} DE &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ &= \sqrt{(6 - 4)^2 + (3 - 4)^2} \\ &= \sqrt{2^2 + (-1)^2} \\ &= \sqrt{5} \end{aligned} \qquad \begin{aligned} AB &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ &= \sqrt{(5 - 1)^2 + (1 - 3)^2} \\ &= \sqrt{4^2 + (-2)^2} \\ &= \sqrt{16 + 4} \\ &= \sqrt{20} \\ &= \sqrt{4 \times 5} \\ &= \sqrt{4} \times \sqrt{5} \\ &= 2\sqrt{5} \end{aligned}$$

Therefore, DE is half the length of AB.